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The coupling of conduction with laminar natural convection along a flat plate

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Abstract—In this paper the entire thermo-fluid-dynamic field resulting from the coupling of natural convection along and conduction inside a heated flat plate is studied by means of two expansions. The first one, describing the field in the lower part of the plate, is a regular series the radius of convergence of which is determined by means of Padé approximant techniques. The second expansion, an asymptotic one, requires a different analysis because of the presence of eigensolutions. The coupling of the two solutions is studied and comparisons with solutions existing in the literature are presented.

1. INTRODUCTION

As POINTED out in ref. [1], when convective heat transfer depends strongly on the thermal boundary conditions, natural convection must be studied as a mixed problem if one needs an accurate analysis of the thermo-fluid-dynamic field. The phenomenon depends on several parameters: therefore in many cases this strong dependence does exist.

In ref. [1] an analysis is given of the relative importance of the parameters of the problem in particular with reference to axial heat conduction. In ref. [2], by extending the analysis of Gosse [3], a technique is shown which improves the results given by the first term of an asymptotic expansion of the solution. In the same paper a new correlation for the evaluation of the heat transfer coefficient is also presented.

This analysis holds for high values of the abscissa x; the value of the point x_0 from which the expansion is valid depends on the parameters that govern the problem.

In this paper we wish to give further contributions to the study of coupled natural convection by evaluating the region which the point x_0 falls in, improving the results concerning the asymptotic expansion by adding terms of higher order with respect to the first one, discussing the general form of the asymptotic expansion, which is singular for the presence of eigensolutions, and determining the expansion holding for small values of x in an accurate way, by evaluating many terms of the series and its radius of convergence by means of Padé approximant techniques.

2. EQUATIONS AND BOUNDARY CONDITIONS

In order to describe the steady two-dimensional flow due to the free convection along a side of a vertical flat plate of thickness b, insulated on the edges and with a temperature T_b maintained on the other side (Fig. 1) one must solve the coupled thermal fields in the solid and in the fluid. The coupling conditions require that the temperature and the heat flux be continuous at the interface.

The temperature T_{so} in the solid is given by

$$T_{so} = T(x,0)-[T_b-T(x,0)]y/b \tag{1}$$

where $T(x,0)$ is the unknown temperature at the interface.

The thermo-fluid-dynamic field in the fluid is governed by the boundary layer equations, which in non-dimensional form may be written as

$$uu_x+vu_y = u_{yy}+\theta;\quad u_x+v_y = 0;\quad u\theta_x+v\theta_y = \theta_{yy}/Pr \tag{2}$$

where u and v are the velocity components, $\theta = (T-T_\infty)/(T_b-T_\infty)$ and Pr is the Prandtl number.

The reference quantities are: $L = \nu^{2/3}/g^{1/3}$ for x, $L/d^{1/4}$ for y and $\nu d^{1/4}$ for the stream function ψ ($u = \psi_y$, $v = -\psi_x$), where $d = (T_b-T_\infty)\beta$.

As the problem of natural convection, for its parabolic character, has no characteristic length, L has been defined in terms of ν and g, which are intrinsic properties of the system.

The reference length along the y-direction has been modified by a factor $d^{-1/4}$ in order to eliminate this quantity from the equations and boundary conditions.

The heat flux continuity condition may be written as

$$\theta(x,0)-1 = p\theta_y(x,0) \tag{3}$$

where

$$p = d^{1/4}b\lambda_f/L\cdot\lambda_s. \tag{4}$$

The boundary conditions that, together with equation (3), must be associated with system (2) are

$$u(x,0) = v(x,0) = u(x,\infty) = \theta(x,\infty) = 0 \tag{5}$$

$$u(0,y) = \theta(0,y) = 0. \tag{6}$$

NOMENCLATURE

b	plate thickness
d	$(T_b - T_\infty)/T_\infty$
f_i	functions occurring in the expansion of ψ
f^*	eigensolution
F_4	function defined by equations (21)
g	acceleration due to gravity
g^*	eigensolution
Gr	Grashof number, gdl^3/v^2
H_i	functions occurring in the expansion of θ
l	length of the plate
L	reference length, $v^{2/3}/g^{1/3}$
m	$px^{-1/4}$
m_1	$x^{1/5}/p^{4/5}$
p	coupling parameter, $b\lambda_f d^{1/4}/L\lambda_s$
Pr	Prandtl number
r	radius of convergence of expansion (14)
T	temperature
T_b	temperature at outside surface of the plate
T_{so}	solid temperature
T_∞	fluid asymptotic temperature
u, v	velocity components
x, y	Cartesian coordinates as indicated in Fig. 1
x_0	coupling abscissa of the two solutions.

Greek symbols

α	constant defined by equations (21)
β	volume thermal expansion coefficient
θ	non-dimensional temperature, $(T-T_\infty)/(T_b-T_\infty)$
θ_i	functions occurring in the expansion of θ
Θ_4	function defined by equations (21)
λ_f, λ_s	fluid and solid thermal conductivities
v	kinematic viscosity
ψ	stream function.

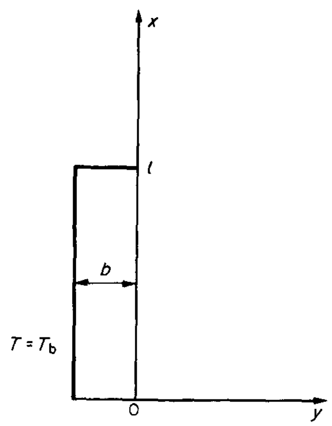

Fig. 1. A vertical flat plate and coordinate system.

3. SOLUTION METHOD

The problem described in the previous section is governed by the coupling parameter p (equation (4)), the order of magnitude of which depends essentially on b/L and λ_f/λ_s, $d^{1/4}$ being of the order of unity. As L is small, b/L attains values much greater than one. When the fluid is air λ_f/λ_s reaches very small values if the plate is highly conductive and reaches the order of 0.1 for materials such as glass. Therefore, p is in many cases, but not always, a small number. An expansion in series based on the smallness of p may be written as follows.

Let $z = y/x^{1/4}$, and $\psi = x^{3/4}f(x,z)$. Then equations (2) and (3) become

$$2f_z^2 - 3ff_{zz} + 4x(f_z f_{zx} - f_x f_{zz}) = 4(\theta + f_{zzz})$$
$$4x(\theta_x f_z - f_x \theta_z) - 3f\theta_z = 4\theta_{zz}/Pr \quad (7)$$

$$\theta(x,0) - 1 = m\theta_z(x,0) \quad (8)$$

where

$$m = px^{-1/4}. \quad (9)$$

Boundary condition (8) suggests us to change variables from x and z to m and z and to expand the functions f and θ in a MacLaurin series with respect to m ($m \to 0$ corresponds to $x \to \infty$) thus writing

$$f = \sum_{i=0}^{\infty} m^i f_i(z)\,; \quad \theta = \sum_{i=0}^{\infty} m^i \theta_i(z). \quad (10)$$

This form of the solution is not satisfactory everywhere because m diverges for vanishing x. Hence this expansion does not hold at $x = 0$ and the initial conditions (6) cannot be satisfied.

Moreover, the linearized problem presents eigenvalues: such a circumstance, although it does not permit the use of an expansion in terms of m of the form of equations (10), enables us to solve the problem of the initial conditions. It is necessary to modify this form and to give boundary conditions at $x = x_0 > 0$, according to equation (6).

To obtain these new initial conditions a different expansion (initial expansion) valid for small values of x will be considered.

Let $s = y/(px)^{1/5}$, $\psi = x^{4/5}g(x,s)/p^{1/5}$, $\theta = x^{1/5}h(x,s)/p^{4/5}$. Then equations (2) may be written as

$$3g_s^2 - 4gg_{ss} + 5x(g_s g_{sx} - g_x g_{ss}) = 5(h + g_{sss})$$
$$hg_s - 4gh_s + 5x(g_s h_x - h_s g_x) = 5h_{ss}/Pr \quad (11)$$

and equation (3) becomes

$$h_s(x,0) = m_1 h(x,0) - 1 \quad (12)$$

where

$$m_1 = x^{1/5}/p^{4/5}. \quad (13)$$

By assuming m_1 and s as independent variables it is possible to expand the functions f and θ in a MacLaurin series with respect to m_1 ($m_1 = 0$ corresponds to $x = 0$)

$$g = \sum_{i=0}^{\infty} m_1^i g_i(s); \quad h = \sum_{i=0}^{\infty} m_1^i h_i(s). \quad (14)$$

In this way if one assumes $g_i'(\infty) = h_i(\infty) = 0$ initial conditions (6) are satisfied as well. Moreover, if x_0 is a point of convergence of expansion (14) it is possible to obtain in this point the initial conditions for a correct expansion in terms of m (asymptotic expansion).

4. EXPANSION FOR SMALL x (INITIAL EXPANSION)

The equations giving the functions g_i and h_i of expansion (14) are

$$5g_i''' - 6g_0'g_i' + 4(g_i g_0'' + g_0 g_i'') - i(g_0'g_i' - g_i g_0'') + 5h_i = M_i$$

$$5h_i''/Pr - h_0 g_i' - h_i g_0' + 4g_0 h_i' + 4g_i h_0' - i(g_0' h_i - h_0' g_i) = N_i \quad (15)$$

where

$$M_i = \sum_{j=1}^{i-1} [3g_j' g_{i-j}' - 4g_j g_{i-j}'' + j(g_j' g_{i-j}' - g_j g_{i-j}'')]$$

$$N_i = \sum_{j=1}^{i-1} [h_j g_{i-j}' - 4g_j h_{i-j}' + j(h_j g_{i-j}' - g_j h_{i-j}')].$$

The boundary conditions are

$$g_i(0) = g_i'(0) = g_i'(\infty) = h_i(\infty) = 0$$

$$h_0'(0) = -1; \quad h_i'(0) = h_{i-1}(0) \quad (i > 0). \quad (16)$$

Equations (15) and (16) represent a standard boundary-value problem which can be easily solved numerically. The only difficulty with this expansion is the evaluation of its radius of convergence $r(Pr)$. Such a function can be obtained by means of the technique of Padé approximants [4].

Padé's idea is to replace a MacLaurin expansion $\Sigma a_n x^n$ of a function $f(x)$ by a sequence of rational functions $P_N^M = P_N(x)/Q_M(x)$ where

$$P_N = \sum_{n=0}^{N} A_n x^n, \quad Q_M = \sum_{n=0}^{M} B_n x^n. \quad (17)$$

Here B_0 may be set equal to 1 without loss of generality. The $M+N+1$ coefficients may be determined so that the equation

$$\sum_{0}^{\infty} a_n x^n = P_N/Q_M$$

is true up to the terms of order n.

Once the B_n coefficients are known it is possible to find the roots of Q_M. The root having the minimum modulus gives an approximation of the radius of convergence of the series.

Padé approximants have been used in the study of many problems. In ref. [5] the impulsive flow past a cylinder is analysed by means of P_{24}^{25}. In ref. [6] the laminar unsteady flow away from a plane stagnation flow is analysed by means of P_{22}^{22}.

We choose the diagonal sequence by assuming $M = N$.

We have checked the reliability of the results by analysing the two expansions related to the wall temperature and to the drag coefficient (i.e. $u_y(x, 0)$), for N varying between 4 and 28 in increments of 2. The two sequences for $Pr = 0.733$ give the following values for the radius of convergence r: 0.97, 1.05, 1.09, 1.11, 1.13, 1.14, 1.15, 1.15, 1.15, 1.16, 1.16, 1.15, for the wall temperature expansion and 1.17, 1.17, 1.16, 1.16, 1.16, 1.17, 1.16, 1.16, 1.15 for the drag coefficient expansion. Both sequences give a value of nearly 1.15 for r. For $Pr = 2.97$ we find $r = 1.65$.

The behaviour of expansion (14) in the range $(0, r)$ of m_1 confirms the value of r found in this way. In Table 1 the values of θ are listed for N terms of the expansion and for $Pr = 2.97$: $m_1 = 1.3$ corresponds to $0.8r$. For small values of m_1 a few terms are sufficient to obtain convergence, as m_1 approaches r the number of terms necessary to reach a good accuracy increases rapidly.

Hence in the range $(0, r)$ for m_1 expansion (14) represents the solution of the problem well.

5. EXPANSION FOR LARGE x (ASYMPTOTIC EXPANSION)

The solution for $m_1 > r$ assumes a form different from that expressed by expansion (10).

Table 1. Values of θ, for $Pr = 2.97$, for several terms of the expansion

	$m_1 = 0.574$	$m_1 = 0.910$	$m_1 = 1.045$	$m_1 = 1.20$	$m_1 = 1.302$	$m_1 = 1.44$	$m_1 = 1.56$
$N = 4$	0.498	0.732	0.905	1.235	1.568	2.25	3.137
$N = 6$	0.490	0.630	0.687	0.780	0.876	1.101	1.441
$N = 8$	0.490	0.618	0.650	0.664	0.650	0.570	0.394
$N = 10$	0.490	0.620	0.657	0.690	0.704	0.703	0.661
$N = 12$	0.490	0.620	0.659	0.699	0.730	0.802	0.946
$N = 14$	0.490	0.620	0.658	0.693	0.712	0.728	0.723
$N = 16$	0.490	0.620	0.658	0.693	0.711	0.717	0.667
$N = 17$	0.490	0.620	0.658	0.693	0.714	0.739	0.762

In fact if one substitutes this expansion into equations (7) one finds at the leading order the following system:

$$2f_0'^2-3f_0f_0''=4(\theta_0+f_0''')$$
$$-3f_0\theta_0'=4\theta_0''/Pr$$
$$f_0(0)=f_0'(0)=f_0'(\infty)=0$$
$$\theta_0(0)=1,\quad \theta_0(\infty)=0 \tag{18}$$

and at the ith order

$$4(f_i'''+\theta_i)-4f_0'f_i'+3f_0f_i''+3f_0''f_i+i(f_i'f_0'-f_if_0'')=S_i$$
$$4\theta_i''/Pr+3f_0\theta_i'+3f_i\theta_0'+i(f_0'\theta_i-\theta_0'f_i)=T_i$$
$$f_i(0)=f_i'(0)=f_i'(\infty)=0$$
$$\theta_i(0)=\theta_{i-1}'(0);\quad \theta_i(\infty)=0 \tag{19}$$

where

$$S_i=\sum_{j=1}^{i-1}[2f_j'f_{i-j}'-3f_jf_{i-j}''-j(f_j'f_{i-j}'-f_jf_{i-j}'')]$$
$$T_i=\sum_{j=1}^{i-1}[-j(\theta_jf_{i-j}'-f_j\theta_{i-j}')-3f_j\theta_{i-j}']. \tag{20}$$

Equations (19) present eigensolutions.

The first one appears in expansion (10) for $i=4$, for linearized equations (19) admit an eigensolution. In fact equations (2) are unchanged by a translation of the origin of the x-axis and their solutions do not change in form when x is substituted for $x-x_0$. If one expands $\psi(x,y,x_0)$ and $\theta(x,y,x_0)$ in powers of x_0, by writing $\psi(x,y,x_0)=x^{3/4}[\psi_0(x,y)+x_0\psi_1(x,y)]$ and $\theta=\theta_0(x,y)+x_0\theta_1(x,y)$, one finds that ψ_1 and θ_1 may be written as $(1/x)F_1(z)$ and $(1/x)G_1(z)$. As this solution holds for any x_0, F_1 and G_1 represent an eigensolution. In particular one finds

$$F_1=(3/4)f_0-(1/4)zf_0';\quad G_1=-(1/4)z\theta_0'$$

where f_0 and θ_0 are given by the leading-order term in expansion (10). Functions F_1 and G_1 satisfy equations (19) with $i=4$ and $S_i=T_i=0$.

Then the first four terms in expansion (10) can be determined by means of equations (19) (the presence of the first eigenvalue does not permit the solution of equations (19) for $i=4$) and the solution can be written in the form

$$f=\sum_{i=0}^{3}m^if_i(z)+m^4Q(m,z)$$
$$\theta=\sum_{i=0}^{3}m^i\theta_i(z)+m^4R(m,z) \tag{10'}$$

where functions $Q(m,z)$ and $R(m,z)$ are not analytic.

Both Q and R may be represented by suitable expansions. To estimate the leading terms of such expansions, say $F_4(m,z)$ and $\Theta_4(m,z)$, we can write equations (10′) as follows:

$$f=\sum_{i=0}^{3}m^if_i(z)+m^4F_4(m,z)+o(m^4)$$
$$\theta=\sum_{i=0}^{3}m^i\theta_i(z)+m^4\Theta_4(m,z)+o(m^4) \tag{10''}$$

where $o(m^4)$ denotes terms of order smaller than m^4.

In order to determine F_4 and Θ_4 we substitute these expressions in equations (7) and (8) and neglect the terms of $o(m^4)$, obtaining for F_4 and Θ_4 the following equations and boundary conditions:

$$4F_{4zzz}+3(F_4f_0''+F_{4zz}f_0)+mf_0'F_{4mz}-(4F_4+mF_{4m})f_0''+4\Theta_4=S_4$$
$$(4/Pr)\Theta_{4zz}+(4\Theta_4+m\Theta_{4m})f_0'-(4F_4+mF_{4m})\theta_0'+3(F_4\theta_0'+\Theta_{4z}f_0)=T_4 \tag{7'}$$
$$\Theta_4(m,0)=\theta_3'(0) \tag{8'}$$

where S_4 and T_4 are defined by equations (20).

These equations may be satisfied by letting

$$F_4(m,z)=f_{4p}(z)+\alpha f_4^*(z)\log m$$
$$\Theta_4(m,z)=\theta_{4p}(z)+\alpha\theta_4^*(z)\log m \tag{21}$$

where $f_4^*(z)$ and $\theta_4^*(z)$ represent a particular eigensolution of system (7′), i.e. satisfy equations (7′) with the right-hand sides S_4 and T_4 vanishing, with the homogeneous boundary conditions $F_4(m,0)=F_{4z}(m,0)=\Theta_4(m,0)=F_{4z}(m,\infty)=\Theta_4(m,\infty)=0$, and α is a constant to be determined later.

In order to calculate f_{4p} and θ_{4p} we write equations (7′) and the pertinent boundary conditions taking into account equations (21) to obtain

$$4f_{4p}'''+4\theta_{4p}+3f_0f_{4p}''-f_0''f_{4p}=S_4+\alpha S_{14}$$
$$4\theta_{4p}''/Pr+\theta_0f_{4p}'-f_0\theta_{4p}'+3\theta_0'f_{4p}=T_4+\alpha T_{14} \tag{22}$$
$$f_{4p}(0)=f_{4p}'(0)=f_{4p}'(\infty)=0 \tag{23}$$
$$\theta_{4p}(0)=\theta_3'(0),\quad \theta_{4p}(\infty)=0 \tag{24}$$

where

$$S_{14}(z)=f_4^{*\prime}f_0'-f_4^*f_0'';\quad T_{14}(z)=\theta_4^*f_0'-f_4^*\theta_0'.$$

It must be noted that the terms containing $\log m$ disappear because f_4^* and θ_4^* are solutions of the equations.

The boundary conditions at infinity may be satisfied by giving suitable values to coefficients C and α in the expressions

$$f_{4p}=Cf_{40}+f_{41}+\alpha f_{42};\quad \theta_{4p}=C\theta_{40}+\theta_{41}+\alpha\theta_{42}$$

where f_{40} and θ_{40} represent a solution of the homogeneous system (22) with $f_{40}(0)=f_{40}'(0)=\theta_{40}(0)=0$ ($f_{40}''(0)$ and $\theta_{40}'(0)$ may be given arbitrary values), f_{41} and θ_{41} and f_{42} and θ_{42} are solutions of equations (22) when the right-hand sides are S_4 and T_4 and S_{14} and T_{14}, respectively, with $f_{4i}(0)=f_{4i}'(0)=\theta_{42}(0)=0$; $\theta_{41}(0)=\theta_3'(0)$ ($f_{4i}''(0)$ and $\theta_{4i}'(0)$ may be given arbitrary values).

In this way we have found two functions F_4 and

Θ_4 of expansion (10″) satisfying all the conditions regarding the y-coordinate.

It must be noted that F_4 and Θ_4 are not completely determined because the functions $F_4^* = F_4 + Af_4^*$ and $\Theta_4^* = \Theta_4 + A\theta_4^*$, where A is an arbitrary constant, also satisfy the system (7′), (8′) and the remaining conditions on the y-coordinate, since f_4^* and θ_4^* are eigensolutions of the problem.

The constant A cannot be determined without knowing all the terms of the expansion of functions $Q(m,z)$ and $R(m,z)$ of which F_4 and Θ_4 represent the leading terms.

Both Q and R may be split in two parts. The first one yielding

$$Q = F_4(m,z) + mF_5(m,z) + \ldots$$

$$R = \Theta_4(m,z) + m\Theta_5(m,z) + \ldots$$

(where F_i and Θ_i contain logarithmic terms in m) enables us to satisfy equations (7) and all the conditions on the y-coordinate, but not the conditions on the x-coordinate which are satisfied only up to third order.

The second one may be obtained from an expansion of f and θ in the eigenfunctions of the problem and enables us to satisfy the conditions on the x-coordinate stemming from the coupling of the asymptotic solution with the initial one at a suitable value of $x = x_0$.

The eigenfunctions have the form $(1/x)^q f_q(z)$, $(1/x)^q\theta_q(z)$, where q is a real parameter which takes on eigenvalues.

The first eigenvalue of q for any Pr is 1; the second one for $Pr = 2.97$ is 2.2, and for $Pr = 0.73$ it is 2.4.

This second expansion can be obtained by standard methods: in the next section we shall give an estimation of constant A appearing in its leading term.

6. RESULTS AND DISCUSSION

The results have shown that the number of terms necessary to represent the initial solution satisfactorily in the range $(0, x_0)$, with x_0 corresponding to a value of m equal to $0.8r$ (r being the radius of convergence), is 17. At $x = x_0$, in the considered case, the asymptotic solution is represented well by means of four terms.

For $Pr = 0.733$, 1.15, 2.97, 7.2, 13.6 the radii of convergence of the initial expansion are 1.16, 1.36, 1.65, 2.03, 2.29, respectively.

The values at $y = 0$ of the first few terms of the initial and asymptotic expansions are listed in Tables 2 and 3.

In order to compare our results with those of ref. [2] we consider a plate with length l and define a Grashof number according to its length; moreover, letting $K = \lambda_s l/\lambda_f b$, we can write $m = Gr^{1/4}/(x/l)^{1/4}K$ and $m_1 = K^{4/5}(x/l)^{1/5}/Gr^{1/5}$. We assume for Gr the value of 10^9.

Table 2. Initial expansion: values of $g_n''(0)$ and $h_n(0)$

	$Pr = 0.733$		$Pr = 2.97$	
n	$g_n''(0)$	$h_n(0)$	$g_n''(0)$	$h_n(0)$
0	1.540	2.042	9.194×10^{-1}	1.411
1	−1.641	−3.083	-6.799×10^{-1}	−1.481
2	1.624	3.789	4.698×10^{-1}	1.271
3	−1.371	−3.886	-2.787×10^{-1}	-9.147×10^{-1}
4	9.453×10^{-1}	3.322	1.360×10^{-1}	5.512×10^{-1}
5	-4.840×10^{-1}	2.298	-4.992×10^{-2}	-2.704×10^{-1}
6	1.210×10^{-1}	1.172	9.470×10^{-3}	9.896×10^{-2}
7	7.296×10^{-2}	-2.853×10^{-1}	3.295×10^{-3}	-1.827×10^{-2}
8	-1.095×10^{-1}	-1.844×10^{-1}	-3.895×10^{-3}	-6.959×10^{-3}
9	5.699×10^{-2}	2.681×10^{-1}	1.570×10^{-3}	7.875×10^{-3}
10	7.548×10^{-3}	-1.632×10^{-1}	2.823×10^{-5}	-3.093×10^{-3}
11	-3.774×10^{-2}	-2.162×10^{-2}	-4.462×10^{-4}	-1.190×10^{-4}
12	2.901×10^{-2}	9.333×10^{-2}	2.770×10^{-4}	9.216×10^{-4}
13	-3.509×10^{-3}	-6.994×10^{-2}	-4.674×10^{-5}	-5.541×10^{-4}
14	-1.454×10^{-2}	7.013×10^{-3}	-5.389×10^{-5}	8.412×10^{-5}
15	1.534×10^{-2}	3.632×10^{-2}	4.985×10^{-5}	1.142×10^{-4}
16	-4.697×10^{-3}	-3.719×10^{-2}	-1.571×10^{-5}	-1.011×10^{-4}
17	-5.632×10^{-3}	1.069×10^{-2}	-5.706×10^{-6}	3.016×10^{-5}

Table 3. Asymptotic expansion: values of $f_n''(0)$ and $\theta_n(0)$

	$Pr = 0.733$		$Pr = 2.97$	
n	$f_n''(0)$	$\theta_n'(0)$	$f_n''(0)$	$\theta_n'(0)$
0	9.532×10^{-1}	-3.591×10^{-1}	7.522×10^{-1}	-5.749×10^{-1}
1	-2.908×10^{-1}	1.315×10^{-1}	-3.693×10^{-1}	3.414×10^{-1}
2	1.143×10^{-1}	-3.593×10^{-2}	2.392×10^{-1}	-1.545×10^{-1}
3	-4.128×10^{-2}	3.845×10^{-8}	-1.515×10^{-1}	8.482×10^{-7}

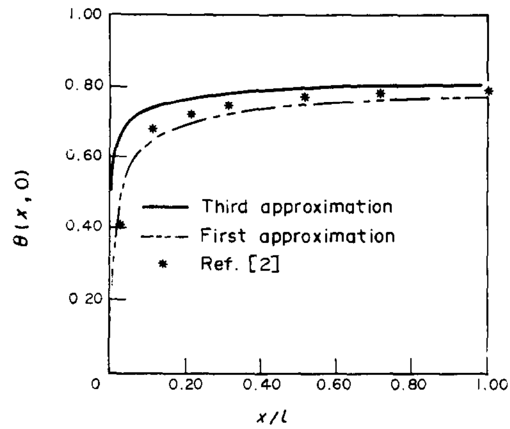

Fig. 2. Non-dimensional temperature at the wall $\theta(x,0)$ in the asymptotic solution for $Pr = 2.97$, $K = 500$, $Gr = 10^9$.

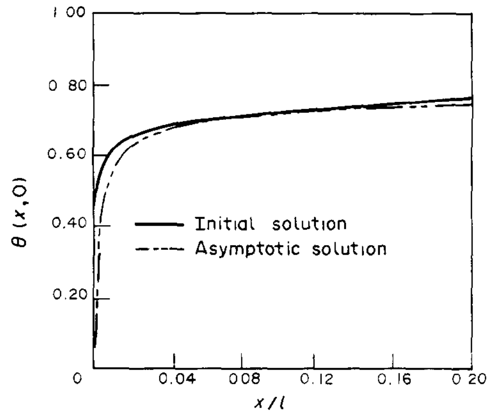

Fig. 4. Non-dimensional temperature at the wall $\theta(x,0)$ for $Pr = 2.97$, $K = 500$.

In Figs. 2 and 3, θ, evaluated by means of the asymptotic expansion, is plotted vs x/l for $K = 500$ and 1000 and $Pr = 2.97$ together with the results of ref. [2]. One can see that the first approximation is not very accurate for the lower value of K: an improvement is obtained by means of the technique of ref. [2]. As K increases, the differences between the first and the third approximation become smaller.

To compare the two expansions we have considered θ and u_y at the wall as a representation of the thermal field and of the fluid-dynamic field. The number of terms of the initial expansion is 17. In Figs. 4 and 5 the case $K = 500$, $Pr = 2.97$ is drawn. In Fig. 4 the dashed curve, corresponding to the asymptotic solution for θ, diverges for $x \to 0$ and differs appreciably from the initial one up to $x/l = 0.05$. For higher values of x/l one sees that the two curves are very close to each other. The value of x/l corresponding to the largest abscissa of convergence of the initial expansion is roughly 0.19; hence in the range (0.05, 0.19) both expansions seem to hold. A similar behaviour is displayed in Fig. 5, but for the considered number of terms the two curves stay very close in a shorter range.

In both cases the value of $x_0/l = 0.06$, corresponding to a value of $m_1 = 0.8r$, is a suitable starting point for the asymptotic expansion.

In Figs. 6 and 7 the curves for the case of $K = 250$ and $Pr = 2.97$ are drawn. For these values $x_0/l > 1$. Therefore, in the whole range (0, 1) the thermo-fluid-dynamic field is governed by the initial solution. The figures show an appreciable difference between the dashed curve (asymptotic solution) and the solid curve (initial solution) for both functions.

In Figs. 8 and 9 the curves relating to $K = 250$ and $Pr = 0.733$ are drawn. In this case $x_0/l = 0.26$. The comparison with Figs. 6 and 7 shows that the lower value of Pr makes the difference between the two solutions very small, except for vanishing values of x.

The previous analysis enables one to obtain the solution of the problem in the entire field using the initial solution for $0 < x < x_0$ and the asymptotic solution for $x > x_0$. x_0 is the starting point of the asymptotic solution and the velocity and temperature profiles obtained from the initial solution represent the initial conditions for the asymptotic solution.

According to the analysis presented in the previous sections the difference between the two solutions at $x = x_0$ must be of the order of magnitude of m^4:

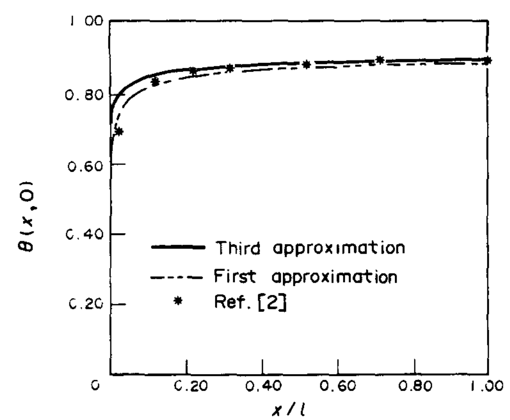

Fig. 3. Non-dimensional temperature at the wall $\theta(x,0)$ in the asymptotic solution for $Pr = 2.97$, $K = 1000$, $Gr = 10^9$.

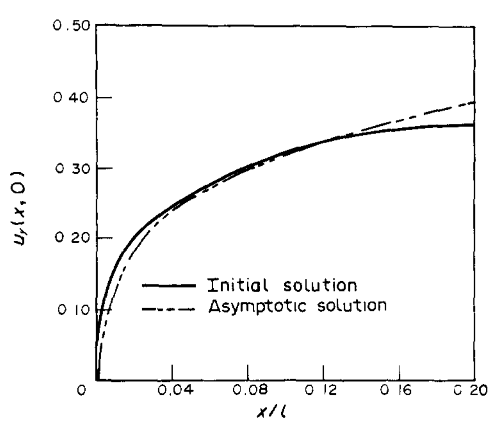

Fig. 5. $u_y(x,0)$ for $Pr = 2.97$, $K = 500$.

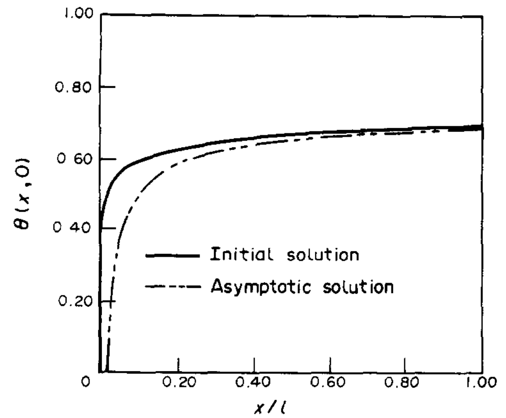

FIG. 6. Non-dimensional temperature at the wall $\theta(x, 0)$ for $Pr = 2.97$, $K = 250$.

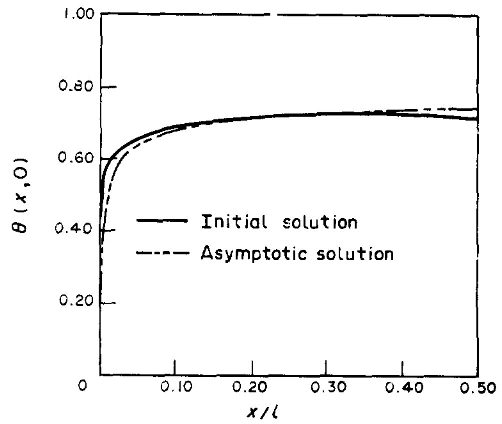

FIG. 8. Non-dimensional temperature at the wall $\theta(x, 0)$ for $Pr = 0.733$, $K = 250$.

in fact we have taken into account the terms of the asymptotic expansion up to the third order. Figures 4–9 confirm such behaviour.

Therefore, when $m(x_0)$ is not too high, as in the considered cases, the approximation in which terms of order m^4 are neglected is satisfactory.

To obtain higher approximations one must consider a new expansion in terms of eigenfunctions, starting from order m^4.

The fourth-order solution can be uniquely determined by the standard procedures that require an analysis of the eigenfunctions.

An idea of the magnitude of the fourth-order terms can be obtained by assuming $\theta(x, 0)$ and $u_y(x, 0)$ as representative of the thermo-fluid-dynamic field. We consider the case $Pr = 2.97$ and $K = 500$, illustrated in Figs. 4 and 5.

Taking into account equations (21) the fourth-order terms may be written in the form $F_4 + Af_4^*$ and $\Theta_4 + A\theta_4^*$, respectively, for functions f and θ, where A is a free constant. As it turns out that, Θ_4 and θ_4^* are vanishing at $y = 0$, one can evaluate A by equating the values of u_y obtained by means of the two solutions at $x = x_0$. Thus one finds for A the value 3.016. In Fig. 10 the initial solution is compared with the asymptotic one, accurate to fourth order. One can see that the two curves are nearly coincident for $0 < x < x_0$.

In this way one obtains that the initial value given to $u_y(x_0, 0)$ in the asymptotic solution, is the exact one while $\theta(x_0, 0)$ assumes in both the third- and fourth-order asymptotic solution the value of 0.706 instead of 0.714.

The differences between the values given by the two solutions for other functions characterizing the profiles, such as

$$\theta_y(x, 0), \quad \int_0^\infty u^2 \, dy, \quad \int_0^\infty u\theta \, dy$$

display the same order of magnitude.

7. CONCLUDING REMARKS

In this paper some aspects of the coupling of conduction inside and laminar convection along a vertical flat plate have been analysed.

As the thermo-fluid-dynamic field for $x \to \infty$ is that of isothermal flat plate flow while for $x \to 0$ it is that of the constant heat flux flat plate flow, two expan-

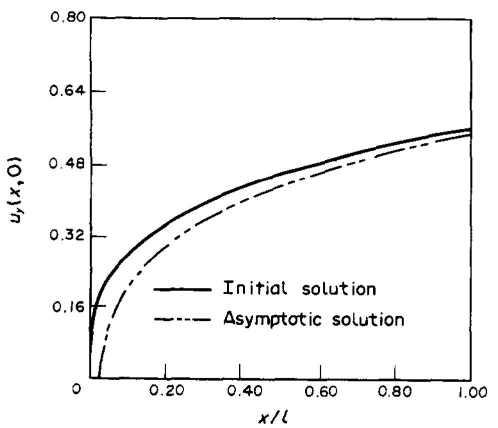

FIG. 7. $u_y(x, 0)$ for $Pr = 2.97$, $K = 250$.

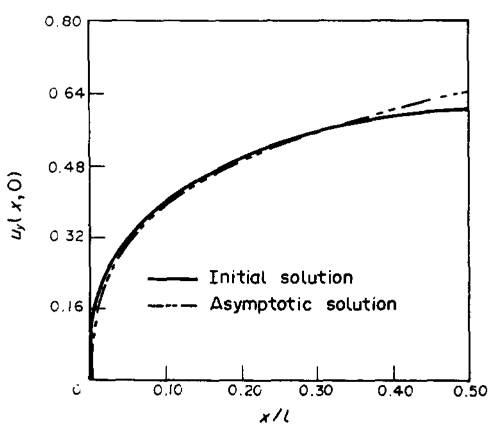

FIG. 9. $u_y(x, 0)$ for $Pr = 0.733$, $K = 250$.

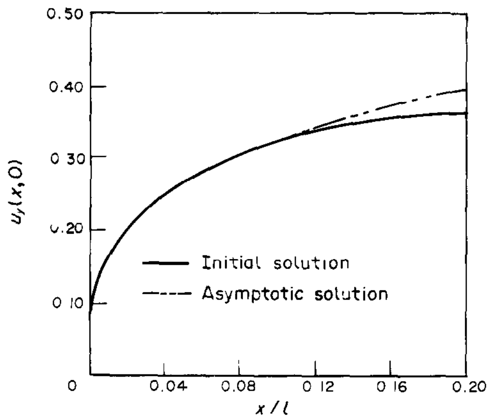

FIG. 10. $u_y(x, 0)$ in the initial solution and in the asymptotic solution (five terms) for $Pr = 2.97$, $K = 500$.

sions (the asymptotic one and the initial one) were considered.

For the initial expansion, with respect to the parameter $m_1 = x^{1/5}/p^{4/5}$, where p is the coupling parameter, it was easy to determine many terms. By means of the Padé approximants it was also possible to evaluate the radius of convergence of such an expansion.

For the asymptotic solution with respect to the parameter $m = p/x^{1/4}$, only four terms can be easily obtained, because the equations for the successive terms of this expansion present eigensolutions, the first of which appears at the fourth order.

It was possible to match the two solutions at a suitable abscissa x_0 by taking into account the first four terms of the asymptotic solution. Moreover, the improvement obtainable by adding the fifth term of the expansion, including the first eigensolution, was estimated.

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COUPLAGE DE LA CONDUCTION ET DE LA CONVECTION NATURELLE LAMINAIRE LE LONG D'UNE PLAQUE PLANE

Résumé—On étudie par la méthode des développements le champ thermofluido-dynamique qui résulte du couplage de la convection naturelle et de la conduction dans une plaque chaude. Le premier développement qui décrit le champ dans la partie inférieure de la plaque est une série régulière dont le rayon de convergence est déterminé au moyen des techniques de Padé. Le second développement de type asymptotique nécessité une analyse différente à cause de la présence des valeurs propres. Le couplage des deux solutions est étudié et on présente des comparaisons avec des solutions déjà existantes.

DIE KOPPLUNG VON WÄRMELEITUNG UND LAMINARER NATÜRLICHER KONVEKTION LÄNGS EINER EBENEN PLATTE

Zusammenfassung—Dieser Beitrag handelt von der Untersuchung des gesamten thermo-fluid-dynamischen Problems, das sich aufgrund der Kopplung von natürlicher Konvektion längs einer beheizten ebenen Platte und der Wärmeleitung in ihrem Innern ergibt. Die erste von zwei Reihenentwicklungen, welche das Problem im unteren Teil der Platte beschreibt, ist eine regelmäßige Reihe, deren Konvergenzradius mittels Pade-Approximations-Techniken bestimmt wurde. Die zweite Entwicklung, eine asymptotische, erfordert aufgrund des Vorhandenseins von Eigenlösungen eine andere Analyse. Es wurde die Kopplung der beiden Lösungen untersucht und Vergleiche mit in der Literatur vorliegenden Lösungen dargelegt.

ВЗАИМНОЕ ВЛИЯНИЕ ТЕПЛОПРОВОДНОСТИ И ЛАМИНАРНОЙ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ ВДОЛЬ ПЛОСКОЙ ПЛАСТИНЫ

Аннотация—С помощью двух разложений исследуется полное термогидродинамическое поле, возникающее в результате связи естественной конвекции вдоль нагретой плоской пластины и теплопроводности в ней. Первое разложение для поля в нижней части пластины представляет собой регулярный ряд, радиус сходимости которого определяется методом аппроксимации Паде. Второе разложение, являющееся асимптотическим, требует иного подхода из-за наличия собственных решений. Исследуется взаимное влияние этих двух решений и дается сравнение с имеющимися в литературе решениями.